

FUgYf'i bhYf'?cbhfc``Y.'Dc`]nY] Xc_i a Ybh]Yfh'nU\`fY]WXY JYf_Y\fgj YfghÒ» Y

9]bgUhnV]`Ubn`XYf`DfcJ]8U!GhfY]ZY.`JYf_Y\fggØbXYf`]b < Ua Vi f['Yfk]gW\h'!`]b_`i g]jY`FUgYfb`i bX Hfi b_Yb\Y]hgZU\fhYb"

8]Y'Dc`]nY]'<Ua Vi f['\Uh'Ua '&&"'i bX'&' "'5i [i gh'&\$&('Y]bY'FY]\Y j cb'J Yf_Y\fg_cbhfc``Yb'Xi fW [YZØ\fhž'X]Y'a Y\fYfY VYa Yf_Ybgk YfhY'9f[YVb]ggY'\Yfj cf[YVfUW\h'\UVYb"'A]h gdYn]Y``Yb'DfcJ]8U!:U\fnYi [Ybž'Ui g[YghUhhYh'ni f'8c_i a YbhUh]cb j cb'J Yf_Y\fgj YfghÒ» Ybž'_cbbhYb'X]Y'6YUa hYb'nU\`fY]W\Y ØVYf\Ò\hY'; YgW\k]bX][_Y]hYb'ZYghghY``Yb'i bX'YbhgdfYW\YbXY A U» bU\a Yb'Y]b`Y]hYb"'Gc`W\Y'9]bgÀhnY'g]bX'HY]``Y]bYf i a ZUggYbXYb'GhfUhY[]Yž'i a 'X]Y'JYf_Y\fgg]W\Yf\Y]h'Ui Z'XYb GhfU» Yb'XYf'<UbgYghUXh'ni 'Yf\Ò\Yb"

? f]h]gWX Y

; $YgWk]bX][_Y]hgØVYfgWkfY]hi b[Yb$

K À\fYbX'Y]bYf'GhfY]ZYbZU\fh'Ui Z'XYf'Df]Y`ghfU» Y']b'< Ua Vi f[!
JYXXY``k i fXYb'X]Y'6YUa hYb'Ui Z'Y]bYb'&\$!'À\f][Yb
A chcffUXZU\fYf'Ui Za Yf_gUa ž'XYf'a]h'Y]bYf'; YgWkk]bX][_Y]h'j cb
-, '_a #\']b'Y]bYf'NcbY'a]h'*\$'_a #\'i bhYfk Y[g'k Uf"'5`g'X]Y'Dc`]nY]

]\b'Ub\]Y`hžghY``hY'g]W\'\YfUi gž'XUgg'Yf'Ui W\'b]W\h']a '6Yg]hn'XYfbchk YbX][Yb': U\fYf`Ui Vb]g'k Ufž'i a 'XUg'A chcffUX'a]h'XYa YbhgdfYW\YbXYb'@Y]ghi b[g[Yk]W\h'ni 'ZU\fYb"'8]YgY'?ca V]bUh]cb Ui g'; YgW\k]bX][_Y]hgØVYfgW\fY]hi b['i bX'i bVYfYW\h][hYa : U\fYb'ZØ\fh'bi b'ni 'Y]bYa 'GhfUZJ YfZU\fYb'[Y[Yb'XYb': U\fYfž'XYfa]h'Y]bYf'; Y`XghfUZYž'Y]bYf'BUW\gW\i `i b['i bX'Y]bYf JYf`Àb[Yfi b['gY]bYf'DfcVYnY]h'fYW\bYb'a i gg"'Ni XYa 'k]fX'Ui W\XYf'<U`hYf'XYg'A chcffUXg'ni f'JYfUbhk cfhi b['[Ync[Ybž'kY]`'Yf XUg': U\fnYi ['Ub'Y]bYb'i bYfZU\fYbYb': U\fYf'ØVYf`UggYb'\Uh"

9]b k Y]hYfYg VYa Yf_Ybgk YfhYg 9fY][b]g ZUbX Ui ZXYf 6i bXYgUi hc VU\b'+']a 'GhY``]b[Ybhi bbY``ghUhh''8cfh`k i fXY Y]b HUI]ZU\fYf VYc VUW\hYhž XYf b]W\h'bi f 'ghUf_' VYgW\`Yi b][hYž gcbXYfb 'Ui W\'a Y\fZUW\'gW\bY```ØVYf\c`h`\Uh''A]h`Y]bYf Xc_i a Ybh]YfhYb'; YgW\k]bX][_Y]h'j cb'%) & _a # 'VY]'Yf`Ui VhYb', \$ _a # 'a i gg XYf &-!\A\f][Y'bi b'a]h'Y]bYa 'gUZh][Yb'6i » [Y`X'j cb &", \$\$'9i fcžni gAhn`]W\Yb'Di b_hYb']a ': U\fY][bi b[gfY[]ghYfi bX Y]bYa 'XfY]a cbUh][Yb': U\fj YfVch'fYW\bYb'' Gc`W\Y'XfUgh]gW\Yb A U» bU\a Yb'gc``Yb'X]Y'J Yf_Y\fghY]`bY\a Yf'ni f'J Yfbi bZh'Vf]b[Yb''

5`_c\c``Ua 'GhYi Yf'i bX']``Y[U`Y'FYbbYb

 $6 YgcbXYfg'U`Ufa]YfYbX'k Uf'XYf'JcfZU```a]hY]bYa '' +! 'Å\f][Yb : U\fYf'Y]bYg': YffUf]'7U`]Zcfb]UZ'XYf'Ua 'gY`VYb'5VYbX'Ui Z'XYf'656 + 'Ui ZZÂ``]['k i fXY''K Å\fYbX'gY]bYf'8 i fW\ZU\fh']a '9`Vhi bbY` k i fXY'Y]bY'; YgW\k]bX][_Y]h'J cb'%-' '_a #\ 'YffY]W\hZ'k Å\fYbX'X]Y Yf'Ui VhY'< OW\gh[YgW\k]bX][_Y]h'VY]', $'_a #\ '`U["'8]Yg'U``Y]b \ÅhhY'VYfY]hg'ni 'Y]bYf'ghfUZfYW\h']W\Yb'JYfZc`[i b['[YZØ\fhZYV\fhZYY]\hY\fY']b' X\X'_Ua 'Y]b'k Y]hYfYf'YfgW\k YfYbXYf'I a ghUbX'\]bni .'8Yf : U\fYf'k i fXY'J YfXÀW\h][hZ'i bhYf'5`_c\c`Y]bZi gg'ni 'ghY\Yb''9]b Xi fW\[YZØ\fhYf'5`_chYghYf[UV'Y]bYb'K Yfh'J cb'$Z'(%'Dfca]``Y'''-\b 5bVYhfUW\h'gY]bYg'U[[fYgg]J Yb': U\fgh]\g'i bX'XYf'\c\Yb ; YgW\k]bX][_Y]h'k i fXY'Y]b'GhfUZJ YfZU\fYb'k Y[Yb'Hfi b_Yb\Y]h']a JYf_Y\fY]bni `Y]hYb'gck]Y'Y]bY'6`i hdfcVY'Ub[YcfXbYh''8Yf : YffUf]'k i fXY'J cf'C fh'VYgW\`U[bU\a hZ'k Ug'X]Y'GW\k YfY'XYf G]hi Uh]cb'j YfXYi h`]W\h''$

8]Y Xi fWX[$YZØ\fhYb$?cbhfc``Yb'i bX'X]Y XUa]h'j YfVi bXYbY

8c_i a YbhUh]cb'j cb'J Yf_Y\fgj YfghÒ» Yb'nY][Yb'_`Ufž'k]Y Yfbgh\UZh'X]Y'Dc`]nY]'<Ua Vi f['X]Y'J Yf_Y\fgg]W\Yf\Y]h'j YfZc`[h" 8i fW\'X]Y'Yj]XYbnVUg]YfhY'8c_i a YbhUh]cbg`U[Y'g]bX'X]Y FYW\hgZc`[Yb'ZØf'X]Y'J Yf_Y\fggØbXYf'_`Uf'dfchc_c``]Yfh'i bX i ba]qqj YfqhÀbX`]W\'bUW\j c``n]Y\VUf"

G]WXYf\Y]h'Ui Z'XYb'GhfU»Yb'Yf\Ò\Yb

8]YgY'9]bgÀhnY'g]bX'Y]b'GW\`ØggY`UgdY_h'Y]bYf'i a ZUggYbXYb

J Yf_Y\fgg]W\Yf\Y]hgghfUhY[]Y"'8]Y'Dc`]nY]'j YfZc`[h'a]h'gc`W\Yb

C dYfUh]cbYb'b]W\h'bi f'X]Y'5\bXi b['j cb'J YfghÒ» Ybž'gcbXYfb

gYhnh'Ui W\'Ui Z'dfÀj Ybh]j Y'A U» bU\a Ybž'i a 'X]Y'G]W\Yf\Y]h'U``Yf

J Yf_Y\fghY]`bY\a Yf'ni '[Yk À\f'Y]ghYb"'Gc`Ub[Y'g]W\'gc`W\Y
; YgW\k]bX][_Y]hgØVYfgW\fY]hi b[Yb'i bX'f]g_UbhYg': U\fj Yf\U`hYb

Ui Z'XYb'GhfU» Yb'nY][Ybž'k]fX'X]Y'Dc`]nY]'k Y]hYf\]b'U_h]j 'V`Y]VYbž
i a 'X]Y'G]W\Yf\Y]h'i bX'XUg'K c\`'U``Yf'6Øf[Yf'ni 'gW\ØhnYb"

9]bgUhn XYf DfcJ]8U!HYW bc`c[]Y]b XYf JYf_Y\fgØVYfk UW i b[

8]Y'DfcJ]8U!HYW\bc`c[]Y'fDfccZ'J]XYc'8UhUŁ'\Uh'X]Y
A O[`]W_Y]hYb'XYf'JYf_Y\fgØVYfkUW\ib['Yf\YV']W\'YfkY]hYfh''
8]YgY'a cXYfbY'HYW\b]_'_ca a h'b]W\h'bi f'VY]'XYf', VYfkUW\ib[
j cb'; YgW\k]bX][_Y]hgØVYfhfYhi b[Yb'ni a '9]bgUhnž'gcbXYfb
Yfa O[`]W\h'Ui W\'X]Y'8c_i a YbhUh]cb'j cb'UbXYfYb
JYf_Y\fgj YfghO\'y Yb']b'9W\hmY]h''8]Y'XUVY]'UbZU``YbXYb'8UhYb
k YfXYb'gck c\``ZØf'X]Y'6Yk Y]gg]W\Yfi b['U`g'Ui W\'ZØf'X]Y
8i fW\ZØ\fi b['j cb'GhfUZ] YfZU\fYb'j Yfk YbXYh''8]Y'n]j]`Yb
9]bgUhn_f\'a\ZhY'XYf'JYf_Y\fgX]fY_h]cb'_ObbYb'Xi fW\'X]Y
JYfk YbXi b['X]YgYf'HYW\bc`c[]Y'df\'an]gY'6Yk Y]gY'gUa a Y`bž'k Ug
ni Y]bYf'VYgqYfYb'JYfi fhY]`i b[qfUhY'ZØ\fh''

8]Y = a d`Ya Ybh]Yfi b['XYf'DfcJ]8U!HYW\bc`c[]Y'k i fXY'U`g

FYU_h]cb'Ui Z'X]Y'k UW\gYbXYb'DfcV`Ya Y'XYf'JYf_Y\fgg]W\Yf\Y]h
]b'i fVUbYb'; YV]YhYb'Ybhk]W_Y`h''8i fW\'X]Y'9fZUggi b['j cb

JYfghÒ»Yb']a 'GhfU»Ybj Yf_Y\f'Xi fW\'\cW\Ui Z`ÒgYbXY'J]XYcXUhYb
_`ObbYb'X]Y'6YUa hYb'gW\bY``Yf'i bX'YZZ]n]YbhYf'Ui Z'[YZÀ\fXYbXY

JYf_Y\fghY]`bY\a Yf`fYU[]YfYb"`8]Yg`hfÀ[h`b]W\h`bi f`ni f FYXi n]Yfi b['j cb'l bZÀ``Ybž'gcbXYfb'Ui W\ni f'6Y_Àa dZi b['j cb FUgYfY]'i bX'5`_c\c``Ua 'GhYi Yf'VY]"'8]Y'Dc`]nY]'<Ua Vi f['n]Y`h XUfUi Z'UVž'Xi fW\gc`W\Y'A U» bU\a Yb'XUg'J YfhfUi Yb'XYf'6Øf[Yf]b'X]Y'JYf_Y\fgØVYfk UW\i b['ni 'ghÀf_Yb"

GhUh]gh]gW\Y'9f\YVib[Yb'ibX']\fY 6YXYihib['ZØf'X]Y'JYf_Y\fgg]W\Yf\Y]h

 $\label{thm:linear} $JYf_Y\setminus fgghUh]gh]_Yb^*]b^*8Yi hgWX^UbX^nY][Yb^*Y]bYb^*U^*Ufa]YfYbXYb $5bgh]Y[^*j_cb^*; YgWXk]bX][_Y]hgØVYfhfYhi b[Yb^*i_bX U^*_c\c^*VYX]b[_hYb^*J_Yf_Y_fgghfUZhUhYb^**@Ui_h^*XYa_^*GhUh]gh]gWXYb $6i_bXYgUa_h^*gdY]ghY^*g]WX^*]a_^*JJ_f^*&$&&^*Yhk_U^*Y]b^*8f]hhY^*XYf $JYf_Y_fgi_bZA^*Y^*Ui_Z^*5i_hcVU_bYb^*Ui_g $$; YgWXk]bX][_Y]hgØVYfhfYhi b[Yb^*8]YgY^*GhUh]gh]_Yb^*j_YfXYi_h^*]WXYb $X]Y^*8f]b[_*]WX_Y]hz^*A_U^*bU_a_Yb^*ni_f^*J_YfVYggYfi_b[_*XYf_JYf_Y_fgg]WXYf_Y]h^*ni_f^*J_YfghAf_Yb^*$

8]Y'DfcJ]8U!9]bgÀhnY'g]bX'Y]b'X]fY_hYg'6Y]gd]Y`'ZØf
XUhYb[Yhf]YVYbY'5bgÀhnY'ni f'JYfVYggYfi b['XYf'G]WXYf\Y]h']a
GhfU»YbjYf_Y\fž'k Ug'a]h'XYb'Vi bXYgkY]hYb'6Ya Ø\i b[Yb'XYf
Dc`]nY]VY\ÒfXYb'ni gUa a YbZÀ``h'''<]Yfni 'nÀ\`h'Ui WX'X]Y'j YfghÀf_hY
5i Z_`Àfi b['ØVYf'X]Y'; YZU\fYb'j cb
; YgWXk]bX][_Y]hgØVYfhfYhi b[Yb'i bX'XYa': U\fYb'i bhYf
5`_c\c`Y]bZi gg"

JYf[`Y]WX ni jYf[Ub[YbYb

Dc`]nY]_Ua dU[bYb

8 Yf '9]bgUhn 'j cb 'HYW bc `c[]Yb 'k]Y 'DfcJ]8 U 'Yf]bbYfh 'Ub 'ZfØ\YfY Dc `]nY]_Ua dU[bYb 'ni f '6Y_Àa dZ b['j cb 'J Yf_Y\fgg]W Yf\Y]hž 'k]Y ni a '6Y]gd]Y `'X]Y '9]bZØ\fi b['j cb'; YgW k]bX][_Y]hg_cbhfc ``Yb']b XYb '&\$\$\$Yf '\$J\fYb" '8 Ua U`gž a]h 'XYf '9]bZØ\fi b['j cb 'ghUh]cbÀfYb i bX 'a cV]`Yb '6`]hnYfUb`U[Ybž_cbbhY 'Y]bY 'g][b]Z]_UbhY 'F YXi _h]cb j cb'; YgW k]bX][_Y]hgØVYfhfYhi b[Yb 'VYcVUW hYh'k YfXYb"

Ÿ\b`]W\ k]Y`\Yi hY`\Uh'Ui W\ X]YgY`ZfØ\Y`A U» bU\a Y`X]Y
ÒZYbh`]W\Y`K U\fbY\a i b[i bX`XUg`J Yf\U`hYb`j cb`5i hcZU\fYfb
VYY]bZi » h"'8]Y`U_hi Y``Y`? Ua dU[bY`ghY\h`YXcW\`Ub`XYf
GW\b]hhghY``Y`nk]gW\Yb`]bbcj Uh]j Yf`HYW\bc`c[]Y`i bX`hfUX]h]cbY``Yf
J Yf_Y\fgØVYfk UW\i b["`K À\fYbX`ZfØ\YfY`GhfUhY[]Yb`cZh`Ui Z`fY]bY
DfÀj Ybh]cb`i bX`dUgg]j Y`A U» bU\a Yb`UVn]Y`hYbž`gYhnYb`U_hi Y``Y
HYW\bc`c[]Yb`k]Y`DfcJ]8U`Ui Z`U_h]j Y`8c_i a YbhUh]cb`i bX`X]fY_hY
♣bhYfU_h]cb`a]h`XYa 'J Yf_Y\fggØbXYf"`

6Y]XY'5bgÀhnY'\UVYb']\fY'Jcf!'ibX'BUW.hY]'YZ'YXcW.'nY][h'X]Y Bihnib['X][]hU'Yf', VYfk UW\ib[ggmghYa YZ'XUgg'XYf'9]bgUhn acXYfbYf'HYW\bc'c[]Yb']b'XYf'JYf_Y\fgØVYfk UW\ib['b]W\h'bif bchk YbX][Z'gcbXYfb'Ui W\'YZZY_h]j']ghz'ia'X]Y'G]W\Yf\Y]h'Ui Z'XYb GhfU» Yb'ni'[Yk À\f'Y]ghYb"

<u>8</u> Y hU]`q

6 Ygi WX Yb 'G]Y 'i bg 'Ui Z. 'X]Y!bUWX f]WX hYb "bYh